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# MOLECULAR DECOMPOSITION AND FOURIER MULTIPLIERS FOR HOLOMORPHIC BESOV AND TRIEBEL-LIZORKIN SPACES

G. CLEANTHOUS†, A. G. GEORGIADIS, AND M. NIELSEN

Abstract. Smooth molecular decompositions for holomorphic Besov and Triebel-Lizorkin spaces on the unit disk of the complex plane are constructed. The decompositions are used to obtain a boundedness result for Fourier multipliers. As further applications, we provide equivalent norms for the spaces under consideration, we consider the implications of the results on Hardy and Hardy-Sobolev spaces, and we study boundedness of coefficient multipliers.

## 1. Introduction

Spaces of holomorphic functions on the unit disk $\mathbb{D}$ of the complex plane play an important role in mathematical analysis, and the study of such spaces naturally connects complex and harmonic analysis. One well known connection arises from the identification of Hardy spaces $H_p(\mathbb{T})$ on the unit circle $\mathbb{T}$ as radial limits of suitable holomorphic functions on the disk. Hardy spaces form an important tool in several applications such as in control and scattering theory.

In the present paper we consider the more general Besov and Triebel-Lizorkin spaces of holomorphic functions on $\mathbb{D}$. The holomorphic Besov and Triebel-Lizorkin spaces are smoothness spaces that generalize many of the important classical holomorphic spaces including Hardy and Hardy-Sobolev spaces, and they allow a unified mathematical treatment. The holomorphic Besov and Triebel-Lizorkin spaces were studied in a systematic way by Oswald [24], while the meromorphic case was considered by Triebel [29]. For more about the study of holomorphic Besov and Triebel-Lizorkin spaces, we refer the reader to [8,13,15]. More generally for the case of several complex variables, a substantial study of holomorphic Triebel-Lizorkin spaces has been done by Ortega and Fáberga in [21, 22]. For the theory of Besov and Triebel-Lizorkin spaces on several geometric contexts see [6,14,16,17,20,25,28] and the references therein.

One major advantage of the general point of view is the possibility of designing stable discrete decomposition systems that work universally for the full range of spaces. Such decomposition systems can then be used to discretise and analyse operators between any two spaces from the family, thus reducing such analysis to a matrix setting, which often makes the analysis more approachable.

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The decomposition approach in Besov and Triebel-Lizorkin spaces was formalised in the seminal papers [9, 10] by Frazer and Jawerth using the $\varphi$-transform framework, and then relying on matrices satisfying a so-called almost-diagonal condition. This approach has been used with great success in various settings to study many types of operators such as Fourier multipliers and pseudo-differential operators, see e.g. [5, 11, 12, 15]. For holomorphic functions on the disk, the corresponding $\varphi$-transform setup based on a periodized version of the Meyer wavelet was introduced by Kyriazis and Petrushev in [15].

One important generalization of the $\varphi$-transform approach is the notion of molecules and associated *molecular decompositions*. Molecules are special families of functions with specific localization and smoothness properties that allow for corresponding stable decompositions on which almost diagonal matrices act naturally to form bounded operators on the associated spaces. Hence, molecules form the foundation for building bounded operators from the class of almost diagonal matrices, and several interesting operators turn out to map molecules to molecules.

Some significant examples of this approach are provided by the boundedness of operators on Triebel-Lizorkin spaces, see Grafakos and Torres [12] and Torres' monograph [27]. Further applications can be found in the recent book of Yuan, Sickel and Yang [30].

The purpose of the present paper is to introduce the notion of molecules in the setting of holomorphic Besov and Triebel-Lizorkin spaces on the disk and apply it to the study of Fourier multipliers. Specifically, the paper contains the following *contributions*:

($\alpha$) Families of smooth synthesis and analysis *molecules* are introduced in Section 3, generalizing the frame families for $F^s_{pq}$ and $B^s_{pq}$ spaces, introduced in [15].

($\beta$) *Molecular decompositions* are obtained in the sense of Frazier and Jawerth [9, 10]; see Theorems 3.8 and 3.12.

($\gamma$) Boundedness of *Fourier multipliers* on $F^s_{pq}$ and $B^s_{pq}$ spaces is obtained in Section 4, based on the molecular decompositions of these spaces.

($\delta$) As *applications equivalent norms* for $F^s_{pq}$ and $B^s_{pq}$ spaces are revisited and the boundedness of Fourier multipliers is adapted to the setting of *coefficient multipliers*.

We would like to mention here that the above results are also *new* in the special cases of *Hardy* and *Hardy-Sobolev* spaces, which are of particular interest for complex analysis. In the final section of the paper, we discuss applications of our results to Hardy and Hardy-Sobolev spaces.

Let us finally mention two issues that we leave for further work. *Harmonic functions* on $\mathbb{R}^n$ form a natural generalisation of holomorphic functions. In the recent preprint [14] Ivanov and Petrushev studied Besov and Triebel-Lizorkin spaces of harmonic functions on the unit ball of $\mathbb{R}^n$. An *open* challenge that we will not pursue here is the extension of our results, and of the almost diagonal machinery of [15], to the case of harmonic functions. Another natural direction to consider is the extension of the results in this paper, as well as the ones of Kyriazis and Petrushev [15], to the case of *several complex variables*, a setup studied in detail in the papers of Ortega and Fábrega [21, 22].

*Notation:* Through the article, positive constants will be denoted by $c$ and they may vary at every occurrence. For every $n \geq 0$, we denote by $\mathcal{C}^n(X)$ the class of $n$-times continuously differentiable functions in a set $X$, by $\mathcal{S}$ the Schwartz functions, and by $\mathcal{C}_0$ the compactly supported continuous functions. The Fourier transform is denoted by $\hat{f}(\xi) := \int_{\mathbb{R}} f(x)e^{-ix\xi}dx$ for $f \in \mathcal{S}$.

## 2. Preliminaries

In this section we present the necessary background for our study. The majority of the tools are coming from [15].

**2.1. Holomorphic Besov and Triebel-Lizorkin spaces.** We denote by $\mathbb{D}$ the unit disk of the complex plane $\mathbb{C}$ and $\mathcal{A} = \mathcal{A}(\mathbb{D})$ the set of analytic functions in $\mathbb{D}$. For every $f \in \mathcal{A}$, and $0 \le r < 1$, we denote

$$\|f(r\cdot)\|_p := \Big(2\pi \int_0^1 |f(re^{2\pi it})|^p dt\Big)^{1/p},\ 0 < p < \infty,$$

and we let

$$\|f(r\cdot)\|_\infty := \sup_{|z|=1} |f(rz)|.$$

Every $f \in \mathcal{A}$ can be expressed as a power series $f(z) = \sum\limits_{n=0}^{\infty} \hat{f}(n)z^n$. We denote

$$J^\alpha f(z) := \sum_{n=0}^{\infty}(n+1)^\alpha \hat{f}(n)z^n,\ \alpha \in \mathbb{R}.$$

The action $J^\alpha f$ is called the $\alpha$-*Weyl derivative* of $f$, when $\alpha > 0$. Specifically, when $\alpha \in \mathbb{N}$, $J^\alpha f = \big[\frac{d}{dz}(z\cdot)\big]^\alpha f$. For more about this operator see [8].

We now define the families of holomorphic Besov and Triebel-Lizorkin spaces. The two families of functions spaces will be central to our study.

**Definition 2.1.** (a) *Let $s \in \mathbb{R}$ and $0 < p, q \le \infty$. Put $\alpha = s + 1$. We say that an analytic $f \in \mathcal{A}$ belongs to the holomorphic Besov space $B^s_{pq} := B^s_{pq}(\mathbb{D})$ if*

$$\|f\|_{B^s_{pq}} := \Big(\int_0^1 (1-r)^{q-1}\|J^\alpha f(r\cdot)\|_p^q dr\Big)^{1/q} < \infty,\ \textit{when } q < \infty$$

*and*

$$\|f\|_{B^s_{p\infty}} := \sup_{0<r<1} (1-r)\|J^\alpha f(r\cdot)\|_p < \infty.$$

(b) *Let $s \in \mathbb{R},\ 0 < p < \infty$ and $0 < q \le \infty$. We say that an analytic $f \in \mathcal{A}$ belongs to the holomorphic Triebel-Lizorkin space $F^s_{pq} := F^s_{pq}(\mathbb{D})$ if*

$$\|f\|_{F^s_{pq}} := \Big\|\Big(\int_0^1 (1-r)^{q-1}|J^\alpha f(r\cdot)|^q dr\Big)^{1/q}\Big\|_p < \infty,\ \textit{when } q < \infty$$

*and*

$$\|f\|_{F^s_{p\infty}} := \Big\|\sup_{0<r<1} (1-r)|J^\alpha f(r\cdot)|\Big\|_p < \infty.$$

**Remark 2.2.** *In Definition 2.1 we may replace the exponent $q-1$ by $(\alpha - s)q - 1$, with $\alpha > s$, and obtain equivalent quasi-norms, see [24]. Below we will use Definition 2.1 as stated.*

2.2. **Distributions.** Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ the unit circle in $\mathbb{C}$. As in [15] we will identify any $f \in \mathcal{A}(\mathbb{D})$ (with some proper growth of Taylor coefficients) with a distribution in $\mathbb{T}$ acting as the boundary values of $f$.

Let $\phi : \mathbb{T} \to \mathbb{C}$ be such that

$$\phi(e^{2\pi ix}) = \sum_{n=0}^{\infty} \hat{\phi}(n) e^{2\pi inx},\ x \in [0,1),$$

where $\hat{\phi}(n) := \int_0^1 \phi(e^{2\pi ix}) e^{-2\pi inx} dx,\ n \geq 0$, are the Fourier coefficients of $\phi$.

We say that $\phi$ is a *test function*, and write $\phi \in \mathcal{D}_+$, when

$$\mathcal{P}_r(\phi) := \sup_{n\geq 0} (n+1)^r |\hat{\phi}(n)| < \infty, \text{ for every } r \geq 0. \tag{2.1}$$

The space of *distributions* $\mathcal{D}'_+$ is the dual of $\mathcal{D}_+$. For $f \in \mathcal{D}'_+$ and $\phi \in \mathcal{D}_+$ we denote the action of $f$ on $\phi$ by $\langle f, \phi \rangle := f(\overline{\phi})$, which is consistent with the inner product on $L^2(\mathbb{T})$,

$$\langle \psi, \phi \rangle := \int_0^1 \psi(e^{2\pi ix}) \overline{\phi(e^{2\pi ix})} dx. \tag{2.2}$$

**Remark 2.3.** *For every $f \in \mathcal{D}'_+$, there exists $r \geq 0$ and $c_r \geq 0$ such that*

$$|\langle f, \phi \rangle| \leq c_r \mathcal{P}_r(\phi), \textit{ for every } \phi \in \mathcal{D}_+.$$

*This gives*

$$|\hat{f}(n)| \leq c_r (n+1)^r,\ n \geq 0, \tag{2.3}$$

*where*

$$\hat{f}(n) := \langle f, x^n \rangle \tag{2.4}$$

*and therefore*

$$f = \sum_{n=0}^{\infty} \hat{f}(n) e^{2\pi inx}, \tag{2.5}$$

*with convergence in $\mathcal{D}'_+$. This means that given $f \in \mathcal{D}'_+$, there exists a holomorphic extension $f \in \mathcal{A}(\mathbb{D})$ such that*

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n,\ z \in \mathbb{D}, \tag{2.6}$$

*where the Taylor coefficients $\hat{f}(n)$ are given by (2.4) and satisfy the polynomial growth (2.3).*

*Conversely, let $f \in \mathcal{A}(\mathbb{D})$ be expressed as (2.6) and the Taylor coefficients $\big(\hat{f}(n)\big)_{n\geq 0}$ satisfy (2.3). Then there exists a unique distribution $f \in \mathcal{D}'_+$ (the boundary value function/distribution of $f$) with Fourier coefficients $\big(\hat{f}(n)\big)_{n\geq 0}$.*

*The considerations above allow us to identify the analytic functions of $\mathcal{A}(\mathbb{D})$ (where Taylor coefficients are known to have polynomial growth (2.3) ) with the distributions of $\mathcal{D}'_+$ viewed as their boundary values.*

*The generalization of the above conversation for harmonic functions on the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$ and distributions on the unit sphere $\mathbb{S}^{n-1}$ can be found in the recent article [14].*

**Notation:** We will write $f(x)$ instead of $f(e^{2\pi ix})$ and we consider $f$ defined on $\mathbb{R}$ and be 1-periodic, so it can be realized as a function defined on $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. By (2.5) we obtain

$$\langle f, \phi \rangle = \sum_{n=0}^{\infty} \hat{f}(n)\overline{\hat{\phi}(n)}, \text{ for every } f \in \mathcal{D}'_+, \ \phi \in \mathcal{D}_+,$$

with the series converging absolutely.

The natural distance on the quotient $\mathbb{T}$ is induced from $\mathbb{R}$ using $|x-y| = \min_{n\in\mathbb{Z}} |x-y+n|$.

Let $\phi \in \mathcal{D}_+$ and $x \in \mathbb{R}$. We define $\tau_x\phi(y) := \phi(y-x)$ and $\widetilde{\phi}(y) := \phi(-y)$ for every $y \in \mathbb{R}$. Let $f \in \mathcal{D}'_+$, the convolution of $f$ with $\phi$ be defined as the test function

$$(f * \phi)(x) := f\big(\tau_x(\widetilde{\phi})\big) = \sum_{n=0}^{\infty} \hat{f}(n)\hat{\phi}(n)e^{2\pi inx}, \ \ x \in [0,1). \tag{2.7}$$

2.3. **An equivalent definition of Besov and Triebel-Lizorkin spaces.** In §2.1 we gave the definition of holomorphic Besov and Triebel-Lizorkin spaces in terms of analytic functions. In §2.2 we explained that we can identify analytic functions (with coefficients of at most polynomial growth) with distributions of $\mathbb{T}$. Based on that fact, we present the following equivalent definition making a natural *link* between complex and harmonic analysis.

Let $\hat{\varphi} \in \mathcal{C}_0^{\infty}[0,2]$ with $\hat{\varphi}(t) \geq 0$ for every $t \in [0,2]$, and $\hat{\varphi}(t) = 1$ for every $t \in [0,1]$. We set $\widehat{\varphi_1}(t) := \hat{\varphi}(t) - \hat{\varphi}(2t)$ and then $\operatorname{supp}\widehat{\varphi_1} \subset [1/2, 2]$.

Let also $\Phi_0(x) := 1$ for every $x$ and

$$\Phi_j(x) := \sum_{\nu=1}^{\infty} \widehat{\varphi_1}(2^{-j+1}\nu)e^{2\pi i\nu x}, \ j \in \mathbb{N}.$$

It can easily be verified that for every $j \in \mathbb{N}$, $\Phi_j(x)$ is the trigonometric polynomial $\sum_{\nu=2^{j-2}}^{2^{j+1}} \widehat{\varphi_1}(2^{-j+1}\nu)e^{2\pi i\nu x}$.

We now recall the following standard definition of Besov and Triebel-Lizorkin spaces on the torus.

**Definition 2.4.** (a) *Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. We say that a distribution $f \in \mathcal{D}'_+$ belongs to the Besov space $B^s_{pq} = B^s_{pq}(\mathbb{T})$ when*

$$\|f\|_{B^s_{pq}} := \left( \sum_{j=0}^{\infty} (2^{sj}\|\Phi_j * f(\cdot)\|_p)^q \right)^{1/q} < \infty$$

*where for $q = \infty$, we use the $\ell^{\infty}$-norm.*

(b) *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. We say that a distribution $f \in \mathcal{D}'_+$ belongs to the Triebel-Lizorkin space $F^s_{pq} = F^s_{pq}(\mathbb{T})$ when*

$$\|f\|_{F^s_{pq}} := \left\| \left( \sum_{j=0}^{\infty} (2^{sj}|\Phi_j * f(\cdot)|)^q \right)^{1/q} \right\|_p < \infty$$

*where for $q = \infty$, we use the $\ell^{\infty}$-norm.*

It has been proved in [24] that Definitions 2.1 and 2.4 coincide. Consequently, Definition (2.4) is independent of the specific functions $\Phi_0, \Phi$ used (up to equivalent norms).

2.4. **Holomorphic Hardy and Hardy-Sobolev spaces.** Let us now present some spaces that play a prominent role in complex analysis which are included in the scale of Triebel-Lizorkin spaces, see for example [4] and the references therein.

**Definition 2.5.** *Let $0 < p \leq \infty$. The Holomorphic Hardy space $H_p = H_p(\mathbb{D})$ is the class of all $f \in \mathcal{A}$ such that*

$$\|f\|_{H_p} := \lim_{r\to 1^-} \|f(r\cdot)\|_p < \infty. \tag{2.8}$$

*Furthermore, let $s \in \mathbb{R}$. The Holomorphic Hardy-Sobolev space $H_p^s = H_p^s(\mathbb{D})$ is the class of all $f \in \mathcal{A}$ such that $J^s f \in H_p$ or*

$$\|f\|_{H_p^s} := \|J^s f\|_{H_p} < \infty. \tag{2.9}$$

**Remark 2.6.** *Let us point out some interesting properties of these spaces:*
*a. Obviously when $s = 0$, we have $H_p^0 = H_p$.*
*b. From [24] we have the identifications*

$$F_{p2}^s \sim H_p^s, \ \textit{for every } \ 0 < p < \infty$$

*In particular, $F_{p2}^0 \sim H_p$ for every $0 < p < \infty$.*

The reader can consult [15, 26] for approximation results in holomorphic Hardy spaces.

2.5. **Wavelet basis.** We now use Meyer's orthonormal bandlimited wavelet basis $\Psi := \{2^{j/2}\psi(2^j x - k),\ j,k \in \mathbb{Z}\}$ on the real line to construct a natural orthonormal system in $L^2(\mathbb{T})$. The system will eventually be used to obtain stable decompositions in the Besov and Triebel-Lizorkin spaces. Recall that the Meyer mother wavelet $\psi : \mathbb{R} \to \mathbb{R}$ satisfies

$$\psi \in \mathcal{S}(\mathbb{R}) \text{ and } \operatorname{supp} \hat{\psi} \subset \{\xi \in \mathbb{R} : 2\pi/3 \leq |\xi| \leq 8\pi/3\}, \tag{2.10}$$

$$\psi(1-x) = \psi(x), \tag{2.11}$$

$$\sum_{j=-\infty}^{\infty} |\hat{\psi}(\xi 2^{-j})|^2 = 1,\ \xi \neq 0. \tag{2.12}$$

It follows from (2.11) that $\psi(x + 1/2)$ is even and we have

$$\hat{\psi}(\xi) = \omega(\xi) e^{-i\xi/2},$$

for the real valued and even function $\omega(\xi) = \widehat{\psi(\cdot + 1/2)}(\xi)$.

We set

$$g_{j,k}(x) := 2^{j/2} \sum_{\ell=-\infty}^{\infty} \psi(2^j(x+\ell) - k),\ 0 \leq k < 2^j,\ j \geq 0,$$

and we observe that $\{1\} \cup \{g_{j,k} : 0 \leq k < 2^j,\ j \geq 0\}$ forms an orthonormal basis for $L^2(\mathbb{T})$. Since $g_{j,k}$ is 1-periodic,

$$g_{j,k}(-x) = g_{j,k}(1-x) = g_{j,k^*}(x)$$

where $k^* := 2^j - k - 1$. Then

$$G := \{1\} \cup \{g_{j,k} + g_{j,k^*} : 0 \le k < 2^{j-1},\ j \ge 0\}$$

is an orthonormal basis for the even functions in $L^2(\mathbb{T})$.

Let us recall the *Poisson's summation formula.* For $f : \mathbb{R} \to \mathbb{C}$,

$$\sum_{n=-\infty}^{\infty} f(x+n) = \sum_{n=-\infty}^{\infty} \hat{f}(2\pi n) e^{2\pi i n x}, \tag{2.13}$$

provided we have some minimal decay and smoothness of $f$, e.g., $|f(t)|, |\hat{f}(t)| \le C(1+|t|)^{-1-\epsilon}$ for some $\epsilon > 0$ will do, see [11].

By (2.13) applied to $\psi(2^j x)$ we obtain

$$g_{j,k}(x) = 2^{-j/2} \sum_{\nu=-\infty}^{\infty} \hat{\psi}(2\pi\nu 2^{-j}) e^{2\pi i \nu (x - k2^{-j})}. \tag{2.14}$$

For every $j \ge 0$ and $0 \le k < 2^j$ we set

$$\begin{aligned} G_{j,k}(x) &:= 2^{-j/2} \sum_{\nu=0}^{\infty} \hat{\psi}(2\pi\nu 2^{-j}) \big( e^{2\pi i \nu (x - k2^{-j})} + e^{2\pi i \nu (x - k^* 2^{-j})} \big) \\ &= 2^{-j/2} \sum_{\nu=0}^{\infty} \omega(2\pi\nu 2^{-j}) \cos\Big( \frac{2\pi\nu}{2^j} \Big( k + \frac{1}{2} \Big) \Big) e^{2\pi i \nu x}. \end{aligned} \tag{2.15}$$

We further set $G_{-1,0}(x) := 1$ and finally

$$\{G_{j,k} : 0 \le k < 2^{j-1},\ j \ge -1\}$$

is orthonormal basis[1] for $H_2$.

2.6. **Discrete decomposition.** We now move towards a stabe discrete decomposition of the Besov and Triebel-Lizorkin spaces. Let $j \ge 1$. For every $0 \le k < 2^{j-1}$ we introduce the dyadic intervals

$$Q = Q_{j,k} := \Big[ \frac{k}{2^j}, \frac{k+1}{2^j} \Big), \tag{2.16}$$

the left end $x_Q := k/2^j$, the length $\ell(Q) := 2^{-j}$ and $Q^* := Q_{j,k^*}$ for $k^* = 2^j - k - 1$. We denote also

$$\mathcal{Q}_j := \{Q_{j,k} : 0 \le k < 2^{j-1}\},\ \mathcal{Q}_j^* := \{Q^* : Q \in \mathcal{Q}_j\} = \{Q_{j,k^*} : 0 \le k < 2^{j-1}\},$$
$$\mathcal{V}_j := \mathcal{Q}_j \cup \mathcal{Q}_j^*,\ \mathcal{Q}_0 = \mathcal{Q}_{-1} := \{[0,1]\},\ \mathcal{Q}_0^* = \mathcal{Q}_{-1}^* := \emptyset$$

and

$$\mathcal{Q} := \bigcup_{j \ge -1} \mathcal{Q}_j,\ \mathcal{V} := \bigcup_{j \ge -1} \mathcal{V}_j.$$

We set for briefness $\mathcal{G} := \{G_Q : Q \in \mathcal{Q}\}$ identifying $Q = Q_{j,k}$. We have that $\mathcal{G}$ is a decomposition system for $\mathcal{D}_+$ and $\mathcal{D}'_+$, i.e., for every $\phi \in \mathcal{D}_+,\ f \in \mathcal{D}'_+$

$$\phi = \sum_{Q \in \mathcal{Q}} \langle \phi, G_Q \rangle G_Q \ \text{ in } \mathcal{D}_+ \tag{2.17}$$

and thus

$$f = \sum_{Q \in \mathcal{Q}} \langle f, G_Q \rangle G_Q \ \text{ in } \mathcal{D}'_+. \tag{2.18}$$

[1]An unconditional basis for $H_p,\ 0 < p < \infty$, and a Schauder basis for $\mathcal{A}(\mathbb{D}) \cup \mathcal{C}^0(\overline{\mathbb{D}})$.

The countable set $\mathcal{Q}$ will be viewed as the domain for the sequences that we will use for the diskrete Besov and Triebel-Lizorkin spaces, which we introduce now.

**Definition 2.7.** (a) *Let $s \in \mathbb{R}$ and $0 < p, q \leq \infty$. The space $b^s_{pq} := b^s_{pq}(\mathcal{Q})$ will contain all the complex valued sequences $a := \{a_Q\}_{Q\in\mathcal{Q}}$ such that*

$$\|a\|_{b^s_{pq}} := \left(\sum_{j=-1}^{\infty} 2^{j(s-1/p+1/2)q}\left(\sum_{Q\in\mathcal{Q}_j}|a_Q|^q\right)^{q/p}\right)^{1/p} < \infty,$$

*using* $\sup_{j\geq -1}$ *when $q = \infty$.*

(b) *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. The space $f^s_{pq} := f^s_{pq}(\mathcal{Q})$ will contain all the complex valued sequences $a := \{a_Q\}_{Q\in\mathcal{Q}}$ such that*

$$\|a\|_{f^s_{pq}} := \left\|\left(\sum_{j=-1}^{\infty} 2^{sjq}\sum_{Q\in\mathcal{Q}_j}[|a_Q|\tilde{\mathbb{1}}_Q(\cdot)]^q\right)^{1/q}\right\|_p < \infty,$$

*where $\tilde{\mathbb{1}}_Q := \ell(Q)^{-1/2}\mathbb{1}_Q$, and using* $\sup_{j\geq -1}$ *when $q = \infty$.*

A fundamental result from [15, Theorem 3.3] is that:

**Theorem 2.8.** *Every $f \in B^s_{pq}$ has a unique representation $f = \sum_{Q\in\mathcal{Q}} c_Q(f)G_Q$ for $c_Q(f) := \langle f, G_Q\rangle$ and*

$$\|f\|_{B^s_{pq}} \sim \|c_Q(f)\|_{b^s_{pq}} \tag{2.19}$$

*and similarly for $F^s_{pq}$.*

## 3. Smooth molecular decomposition

In this section we extend Theorem 2.8 by replacing $\{G_Q\}_{Q\in\mathcal{Q}}$ by more general families, the so-called families of smooth molecules that we will introduce below. The fundamental tool to obtain such decomposition is notion of almost diagonal operators first considered by Frazer and Jawerth [9, 10] and later adapted to the holomorphic setting by Kyriazis and Petrushev in [15]. We recall the definition in [15].

### 3.1. Almost diagonal operators.

A linear operator $A$, acting on $b^s_{pq}$ or $f^s_{pq}$ with matrix $(\alpha_{QP})_{Q,P\in\mathcal{Q}}$, is called almost-diagonal if there exists $\varepsilon > 0$ such that

$$\sup_{Q,P\in\mathcal{Q}} |\alpha_{QP}|/\omega_\varepsilon(Q,P) < \infty \tag{3.20}$$

where

$$\begin{aligned}\omega_\varepsilon(Q,P) :=& 2^{(i-j)s}\min\left\{2^{(i-j)(1+\varepsilon)/2}, 2^{(j-i)(\mathcal{J}+\varepsilon/2-1/2)}\right\}\\ &\times\left(\left(1+2^{\min(i,j)}|x_Q-x_P|\right)^{-\mathcal{J}-\varepsilon}+\left(1+2^{\min(i,j)}|x_Q-x_{P^*}|\right)^{-\mathcal{J}-\varepsilon}\right)\end{aligned}$$

for $Q \in \mathcal{Q}_j,\ P \in \mathcal{Q}_i,\ j, i \geq -1$ and $\mathcal{J} := 1/\min(1,p)$ for $b^s_{pq}$ and $\mathcal{J} := 1/\min(1,p,q)$ for $f^s_{pq}$.

The importance of this class is emphasised by the following result from [15].

**Proposition 3.1.** *Let $s \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. An almost diagonal operator on $f^s_{pq}$ is bounded.*

For the analysis below, we will need the following three well-known estimates:

**Lemma 3.2.** [15, p. 453]. *Let $h \in \mathcal{C}^2(\mathbb{R}),\ N > 1$ and $M > N+1$. If*

$$\int_{\mathbb{R}} x^\gamma h(x)dx = 0, \ \textit{ for every } \ 0 \le \gamma \le N,$$

$$|h^{(\gamma)}(x)| \le A(1+|x|)^{-M}, \ \textit{ for every } \ \gamma = 0,1,2,$$

*then there exists a constant $c > 0$:*

$$\left| \sum_{\nu=0}^{\infty} \hat{h}(2\pi\nu n^{-1}) e^{2\pi i \nu x} \right| \le cAn(1+n|x|)^{-N}, \ n \ge 1, \ |x| \le \frac{1}{2}.$$

In the next two Lemmata, for any function $h$ we denote $h_k(x) := 2^k h(2^k x),\ k \in \mathbb{Z}$.

**Lemma 3.3.** [15, p. 454]. *Let $g \in \mathcal{C}^N(\mathbb{R}),\ h \in \mathcal{C}(\mathbb{R})$ such that*

$$|g^{(\gamma)}(x)| \le A_1(1+|x|)^{-M_1}, \ \ 0 \le \gamma \le N,$$

$$|h(x)| \le A_2(1+|x|)^{-M_2}$$

*and*

$$\int_{\mathbb{R}} x^\gamma h(x)dx = 0, \ \ 0 \le \gamma \le N-1,$$

*for $N \ge 1,\ M_2 \ge M_1,\ M_2 > N+1$. Then for every $k \ge 0$*

$$|g * h_k(x)| \le cA_1A_2 2^{-kN}(1+|x|)^{-M_1}.$$

We will also need the following estimate.

**Lemma 3.4.** [10, p. 152]. *Let $M > 1,\ A, B > 0$ and*

$$|g(x)| \le A(1+|x|)^{-M}, \ |h(x)| \le B(1+|x|)^{-M}.$$

*Then there exists a constant $c = c_M > 0$ such that for every $k \ge 0$*

$$|g * h_k(x)| \le cAB(1+|x|)^{-M}.$$

3.2. **Smooth synthesis molecules.** We now introduce the families of smooth synthesis molecules. Such families for the case of $\mathbb{R}^n$ appear by Frazier and Jawerth in [9,10], and in the anisotropic setting by Bownik [1] and Bownik-Ho [2]. We define families of smooth synthesis molecules for holomorphic Triebel-Lizorkin spaces.

**Definition 3.5.** *Let $s \in \mathbb{R},\ p \in (0,\infty)$ and $q \in (0,\infty]$, and recall that $\mathcal{J} = 1/\min(1,p,q)$. We further introduce the parameters*

$$N := [\mathcal{J} - s] \textit{ when } \mathcal{J} - s - 1 \ge 0, \ \textit{ and } \ N := 0 \textit{ when } \mathcal{J} - s - 1 < 0,$$

$$K := [s] + 1 \textit{ when } s \ge 0, \ \textit{ and } \ K := 0 \textit{ when } s < 0.$$

*Let $m \in \mathcal{C}^{K+2}(\mathbb{R})$ be chosen such that there exists*

$$\mathcal{M} > [\mathcal{J}] + 2$$

*for which:*
(i)

$$|m^{(\gamma)}(x)| \le (1+|x|)^{-\mathcal{M}}, \ 0 \le \gamma \le 2. \tag{3.21}$$

(ii) *If $s \ge 0$ then*

$$|m^{(\gamma)}(x)| \le (1+|x|)^{-\max(\mathcal{M}, s+3)}, \ 0 \le \gamma \le K+2. \tag{3.22}$$

(iii) *If $\mathcal{J}-s-1\geq 0$ then*

$$\int_{\mathbb{R}} x^\gamma m(x)dx=0,\ 0\leq\gamma\leq N-1 \tag{3.23}$$

*and*

$$|m(x)|\leq(1+|x|)^{-\max(\mathcal{M},\mathcal{J}-s+2)}. \tag{3.24}$$

*For every $Q\in\mathcal{Q}_j$, we set*

$$M_Q(x):=2^{-j/2}\sum_{\nu=0}^{\infty}\hat{m}(2\pi\nu 2^{-j})\big(e^{2\pi i\nu(x-x_Q)}+e^{2\pi i\nu(x-x_{Q^*})}\big),\ \textit{when } j\geq 0 \tag{3.25}$$

*and $M_Q(x):=1$ when $j=-1$.*
*Then we say that $\{M_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth synthesis molecules for $F^s_{pq}$.*

The corresponding definition for the case of Besov spaces is:

**Definition 3.6.** *Let $s\in\mathbb{R},\ p,q\in(0,\infty]$ and put $\mathcal{J}:=1/\min(1,p)$. If the function $m\in\mathcal{C}^{K+2}(\mathbb{R})$ satisfies (3.21)-(3.24), then we will say that the family $\{M_Q\}_{Q\in\mathcal{Q}}$ as in (3.25) is a family of smooth synthesis molecules for $B^s_{pq}$.*

**Remark 3.7.** *The family $\{G_Q\}_{Q\in\mathcal{Q}}$ is a constant multiple of a family of smooth synthesis molecules for both $F^s_{pq}$ and $B^s_{pq}$ for every triple of indices $(s,p,q)$.*

One of the central purposes of this article is to generalize Theorem 2.8 to the following

**Theorem 3.8.** *(Smooth molecular synthesis). Let $s\in\mathbb{R},\ p\in(0,\infty)$ and $q\in(0,\infty]$. Then there exists a constant $c>0$ such that if $\{M_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth synthesis molecules for $F^s_{pq}$, then for every $f=\sum_{Q\in\mathcal{Q}}a_QM_Q$ it holds that*

$$\|f\|_{F^s_{pq}}\leq c\|a_Q\|_{f^s_{pq}},\ \textit{for all } \{a_Q\}\in f^s_{pq}.$$

For the proof of Theorem 3.8, we will need the following lemma, which should be of independent interest. Let us first recall that: For every $j\geq 0$ and $0\leq k<2^j$

$$G_{j,k}(x)=2^{-j/2}\sum_{\nu=0}^{\infty}\hat{\psi}(2\pi\nu 2^{-j})\big(e^{2\pi i\nu(x-k2^{-j})}+e^{2\pi i\nu(x-k^*2^{-j})}\big)$$

and $G_{-1,0}(x)=1$.

**Lemma 3.9.** *Let $s\in\mathbb{R},\ p\in(0,\infty)$ (or $p\in(0,\infty]$) and $q\in(0,\infty]$ and $\{M_Q\}_{Q\in\mathcal{Q}}$ be a family of smooth synthesis molecules for $F^s_{pq}$ (or $B^s_{pq}$). Then the operator $A$ given by the matrix*

$$a_{QP}:=\langle M_P,G_Q\rangle,\ Q,P\in\mathcal{Q}$$

*is almost diagonal.*

*Proof.* Let $Q = Q_{j,k}$ and $P = P_{\mu,\ell}$ for some $j, \mu \geq 0$, and $k = 0, 1, \ldots, 2^{j-1} - 1$, $\ell = 0, 1, \ldots, 2^{\mu-1} - 1$. Then by (3.25)

$$\begin{aligned}
a_{QP} = \langle M_P, G_Q \rangle &= 2^{-(\mu+j)/2} \sum_{\nu=0}^{\infty} \hat{m}(2\pi\nu 2^{-\mu}) \overline{\hat{\psi}(2\pi\nu 2^{-j})} e^{2\pi i\nu(x_P - x_Q)} \\
&+ 2^{-(\mu+j)/2} \sum_{\nu=0}^{\infty} \hat{m}(2\pi\nu 2^{-\mu}) \overline{\hat{\psi}(2\pi\nu 2^{-j})} e^{2\pi i\nu(x_{P^*} - x_Q)} \\
&+ 2^{-(\mu+j)/2} \sum_{\nu=0}^{\infty} \hat{m}(2\pi\nu 2^{-\mu}) \overline{\hat{\psi}(2\pi\nu 2^{-j})} e^{2\pi i\nu(x_P - x_{Q^*})} \\
&+ 2^{-(\mu+j)/2} \sum_{\nu=0}^{\infty} \hat{m}(2\pi\nu 2^{-\mu}) \overline{\hat{\psi}(2\pi\nu 2^{-j})} e^{2\pi i\nu(x_{P^*} - x_{Q^*})} \\
&=: a_1 + \cdots + a_4. \qquad (3.26)
\end{aligned}$$

We first consider the case $\mu \geq j$. For convenience, we note that

$$\hat{m}(2\pi\nu 2^{-\mu}) \overline{\hat{\psi}(2\pi\nu 2^{-j})} = \widehat{m_{\mu-j} * \tilde{\psi}}(2\pi\nu 2^{-j}) \qquad (3.27)$$

where $\tilde{\psi}(x) = \overline{\psi(-x)}$.

We study separately the cases when $\mathcal{J} - s - 1 < 0$ and $\mathcal{J} - s - 1 \geq 0$.

Case $\alpha$: $\mathcal{J} - s - 1 < 0$. Let $r \in \{0, 1, 2\}$. Clearly,

$$\Big(m_{\mu-j} * \tilde{\psi}\Big)^{(r)}(x) = \Big(m_{\mu-j} * \tilde{\psi}^{(r)}\Big)(x). \qquad (3.28)$$

We will apply Lemma 3.4 to the functions $\tilde{\psi}^{(r)}$ and $m$ (playing the roles of $g$ and $h$ respectively).

By (3.21)

$$|m(x)| \leq (1 + |x|)^{-\mathcal{M}}, \text{ for } \mathcal{M} > [\mathcal{J}] + 2. \qquad (3.29)$$

By (2.10), and standard arguments, we see that (3.29) holds true for $\tilde{\psi}^{(r)}$. Thus, by Lemma 3.4 and (3.28),

$$\left| \Big(m_{\mu-j} * \tilde{\psi}\Big)^{(r)}(x) \right| \leq c(1 + |x|)^{-\mathcal{M}}. \qquad (3.30)$$

By (2.10), we have

$$\int_{\mathbb{R}} x^{\gamma} \big(m_{\mu-j} * \tilde{\psi}\big)(x) dx = 0, \ 0 \leq \gamma \leq [\mathcal{J}] + 1. \qquad (3.31)$$

Since $\mathcal{M} > [\mathcal{J}] + 2$, and thanks to (3.30) and (3.31), we apply Lemma 3.2 to the function $m_{\mu-j} * \tilde{\psi}$. Then by (3.27)

$$\begin{aligned}
|a_1| &= 2^{-(\mu+j)/2} \left| \sum_{\nu=0}^{\infty} \widehat{m_{\mu-j} * \tilde{\psi}}(2\pi\nu 2^{-j}) e^{2\pi i\nu(x_P - x_Q)} \right| \\
&\leq c 2^{-(\mu+j)/2} 2^j (1 + 2^j |x_P - x_Q|)^{-[\mathcal{J}]-1} \\
&= c 2^{-(\mu-j)/2} (1 + 2^j |x_P - x_Q|)^{-[\mathcal{J}]-1} \\
&\leq c 2^{-(\mu-j)(\mathcal{J}-s-1+\frac{\varepsilon}{2}+\frac{1}{2})} (1 + 2^j |x_P - x_Q|)^{-\mathcal{J}-\varepsilon} \\
&\leq c\omega_{\varepsilon}(Q, P),
\end{aligned}$$

where we chose $0 < \varepsilon \le \min\big(2(-\mathcal{J}+s+1), [\mathcal{J}]+1-\mathcal{J}\big)$.

Case $\beta$: $\mathcal{J}-s-1 \ge 0$. In this case $N = [\mathcal{J}-s] \ge 1$. By (3.23), we have the vanishing moments

$$\int_{\mathbb{R}} x^\gamma m(x)dx = 0,\ 0 \le \gamma \le N-1.$$

On the other hand, by (3.24),

$$|m(x)| \le (1+|x|)^{-\max(\mathcal{M},\mathcal{J}-s+2)},$$

and the same holds true for $\tilde{\psi}^{(\gamma)}$ for every $\gamma \ge 0$. Then we can apply Lemma 3.3 (substituting $\tilde{\psi}^{(r)} = g,\ m = h$) and we obtain by (3.28)

$$\left|\Big(m_{\mu-j} * \psi\Big)^{(r)}\right| \le c2^{-(\mu-j)N}(1+|x|)^{-\max(\mathcal{M},\mathcal{J}-s+2)},\ r = 0,1,2.$$

On the other hand, because of (2.10), the function $m_{\mu-j} * \psi$ has vanish moments of any order. Hence, Lemma 3.2 implies that

$$\begin{aligned}|a_1| &\le c2^{-(\mu+j)/2}2^{-(\mu-j)N}2^j(1+2^j|x_P-x_Q|)^{-[\mathcal{J}]-1}\\ &\le c\omega_\varepsilon(Q,P),\end{aligned}$$

for any $0 < \varepsilon \le \min\big(2(N-\mathcal{J}+s+1), [\mathcal{J}]+1-\mathcal{J}\big)$.

Therefore, we have proved that there exist constants $c > 0$ and $\varepsilon > 0$ such that

$$|a_1| \le c\omega_\varepsilon(Q,P)$$

when $\mu \ge j$.

Now we consider the case $\mu \le j$. We have the expression

$$\hat{m}(2\pi\nu 2^{-\mu})\overline{\hat{\psi}(2\pi\nu 2^{-j})} = \widehat{m * \tilde{\psi}_{j-\mu}}(2\pi\nu 2^{-\mu}), \tag{3.32}$$

and for $r \in \{0,1,2\}$,

$$\Big(m * \tilde{\psi}_{j-\mu}\Big)^{(r)}(x) = \Big(m^{(r)} * \tilde{\psi}_{j-\mu}\Big)(x). \tag{3.33}$$

This time two cases present themselves: $s < 0$ and $s \ge 0$.

Case $\gamma$: $s < 0$. Arguing as in Case $\alpha$, and using (3.33), it follows that

$$\left|\Big(m * \tilde{\psi}_{j-\mu}\Big)^{(r)}(x)\right| \le c(1+|x|)^{-\mathcal{M}},\ r = 0,1,2.$$

For every $0 \le \gamma \le [\mathcal{J}]+1$, (2.10) implies that

$$\int_{\mathbb{R}} x^\gamma\big(m * \tilde{\psi}_{j-\mu}\big)(x)dx = 0.$$

An application of Lemma 3.2 to the function $m * \tilde{\psi}_{j-\mu}$ leads to

$$\begin{aligned}|a_1| &\le c2^{-(\mu+j)/2}2^\mu(1+2^\mu|x_P-x_Q|)^{-[\mathcal{J}]-1}\\ &\le c\omega_\varepsilon(Q,P)\end{aligned}$$

for any $0 < \varepsilon \le \min(-2s, [\mathcal{J}]+1-\mathcal{J})$.

Case $\delta$: $s \ge 0$. Now $K = [s]+1 \ge 1$ and (3.22) gives

$$\left|m^{(r+\gamma)}(x)\right| \le c(1+|x|)^{-\max(\mathcal{M},s+3)},\ 0 \le \gamma \le K. \tag{3.34}$$

Since $\max(\mathcal{M}, s+3) > K+1$, from Lemma 3.3 and (3.33) we derive

$$\left|\left(m * \tilde{\psi}_{j-\mu}\right)^{(r)}(x)\right| \leq c2^{-(j-\mu)K}(1+|x|)^{-\max(\mathcal{M},s+3)},\ r=0,1,2.$$

Now, bearing in mind that $\max(\mathcal{M}, s+3) > [\mathcal{J}]+2$, we apply Lemma 3.2 and obtain

$$\begin{aligned}|a_1| &\leq c2^{-(\mu+j)/2}2^{-(j-\mu)K}2^{\mu}(1+2^{\mu}|x_P-x_Q|)^{-\mathcal{M}}\\ &= c2^{-(j-\mu)(K+\frac{1}{2})}(1+2^{\mu}|x_P-x_Q|)^{-\mathcal{M}}\\ &\leq c\omega_\varepsilon(Q,P)\end{aligned}$$

for any $0<\varepsilon \leq \min\big(2(K-s), \mathcal{M}-\mathcal{J}\big)$.

Cases $\alpha - \delta$ combined give that there exist constants $c, \varepsilon > 0$ such that

$$|a_1| \leq c\omega_\varepsilon(Q,P),\ \text{when } Q,P \notin \mathcal{Q}_{-1}. \tag{3.35}$$

Proceeding in a similar way, we obtain

$$|a_2| \leq c\omega_\varepsilon(Q,P). \tag{3.36}$$

For the terms $a_3$ and $a_4$, we need some additional considerations. Let us focus on $a_3$. Assume $\mu \leq j$. With the same strategy as in the estimation of $a_1$, we extract terms of the form

$$(1+2^{\mu}|x_P - x_{Q^*}|)^{-\mathcal{J}-\varepsilon}.$$

It then holds that

$$\begin{aligned}1+2^{\mu}|x_Q-x_{P^*}| &= 1+2^{\mu}\left|\frac{k}{2^j}-1+\frac{\ell}{2^{\mu}}+\frac{1}{2^{\mu}}\right|\\ &\leq 1+2^{\mu}\left|\frac{k}{2^j}-1+\frac{1}{2^j}+\frac{\ell}{2^{\mu}}\right|+2^{\mu}\left|\frac{1}{2^{\mu}}-\frac{1}{2^j}\right|\\ &\leq 2\big(1+2^{\mu}|x_{Q^*}-x_P|\big),\end{aligned}$$

so $\big(1+2^{\mu}|x_{Q^*}-x_P|\big)^{-\mathcal{J}-\varepsilon} \leq c\big(1+2^{\mu}|x_Q-x_{P^*}|\big)^{-\mathcal{J}-\varepsilon}$. The last estimate leads us to

$$|a_3| \leq c\omega_\varepsilon(Q,P), \tag{3.37}$$

and in a similar way to

$$|a_4| \leq c\omega_\varepsilon(Q,P). \tag{3.38}$$

By a combination of (3.26) with (3.35)-(3.38), we conclude that there exist constants $c, \varepsilon > 0$ such that

$$|a_{QP}| \leq c\omega_\varepsilon(Q,P),\ \text{for every } Q,P \in \mathcal{Q}\setminus\mathcal{Q}_{-1}. \tag{3.39}$$

Let us now allow $Q, P \in \mathcal{Q}_{-1}$. We assume that $\mu > j = -1$; the other cases are similar.

Since $j=-1$, $G_{-1,0}(x)=1$, $Q=[0,1]$ and $x_Q=0$. Then $|a_{QP}| = 2^{-\mu/2}|\hat{m}(0)|$ and

$$\begin{aligned}\omega_\varepsilon(Q,P) &= c2^{-\mu(\mathcal{J}-s+\frac{\varepsilon}{2}-\frac{1}{2})}\Big((1+2^{-1}|x_P|)^{-\mathcal{J}-\varepsilon}+(1+2^{-1}|x_{P^*}|)^{-\mathcal{J}-\varepsilon}\Big)\\ &\geq c2^{-\mu(\mathcal{J}-s+\frac{\varepsilon}{2}-\frac{1}{2})}.\end{aligned}$$

- If $\mathcal{J}-s-1 \geq 0$, by (3.23), $\hat{m}(0)=0$, so $|a_{QP}| = 0 \leq c\omega_\varepsilon(Q,P)$.

• If $\mathcal{J} - s - 1 < 0$, then

$$|a_{QP}| \leq c2^{-\mu/2} \leq c2^{-\mu(\mathcal{J}-s+\frac{\varepsilon}{2}-\frac{1}{2})} \leq c\omega_\varepsilon(Q,P)$$

for any $0 < \varepsilon \leq 2(-\mathcal{J} + s + 1)$.

In other words, (3.39) holds true for every $Q, P \in \mathcal{Q}$ and the proof is complete. □

We are now ready to prove Theorem 3.8.

*Proof.* Let $f = \sum_{Q\in\mathcal{Q}} a_Q M_Q$ for $(a_Q) \in f^s_{pq}$, and let $(M_Q)$ be a family of smooth synthesis molecules for $F^s_{pq}$. From (2.18)

$$M_P = \sum_{Q\in\mathcal{Q}} \langle M_P, G_Q\rangle G_Q, \text{ for every } P \in \mathcal{Q} \text{ (in } \mathcal{D}'_+).$$

Let $A$ be the operator acting on $f^s_{pq}$ with matrix

$$a_{QP} := \langle M_P, G_Q\rangle,\ Q, P \in \mathcal{Q}.$$

Then we have

$$\begin{aligned} f = \sum_{P\in\mathcal{Q}} a_P M_P &= \sum_{P\in\mathcal{Q}} a_P \sum_{Q\in\mathcal{Q}} a_{QP} G_Q \\ &= \sum_{Q\in\mathcal{Q}} \left(\sum_{P\in\mathcal{Q}} a_{QP} a_P\right) G_Q = \sum_{Q\in\mathcal{Q}} (Aa)_Q G_Q. \end{aligned}$$

By (2.19) we have

$$\|f\|_{F^s_{pq}} = \Big\| \sum_{Q\in\mathcal{Q}} (Aa)_Q G_Q \Big\|_{F^s_{pq}} \leq c\|Aa\|_{f^s_{pq}} \leq c\|a\| f^s_{pq},$$

where for the last inequality we used that $A$ is almost diagonal thanks to Lemma 3.9, and hence it is bounded on $f^s_{pq}$ by Proposition 3.1. This concludes the proof. □

3.3. **Smooth analysis molecules.** We introduce a new family which might be considered as a "dual" to the family of smooth synthesis molecules.

**Definition 3.10.** *We fix $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Let $b \in \mathcal{C}^{N+2}(\mathbb{R})$ be such that there exists $\mathcal{M} > [\mathcal{J}] + 2$ :*
(i)

$$|b^{(\gamma)}(x)| \leq (1 + |x|)^{-\mathcal{M}},\ 0 \leq \gamma \leq 2. \tag{3.40}$$

(ii) *If $\mathcal{J} - s - 1 \geq 0$ then*

$$|b^{(\gamma)}(x)| \leq (1 + |x|)^{-\max(\mathcal{M},\mathcal{J}-s+2)},\ 0 \leq \gamma \leq N + 2. \tag{3.41}$$

(iii) *If $s \geq 0$ then*

$$\int_{\mathbb{R}} x^\gamma b(x)dx = 0,\ 0 \leq \gamma \leq [s], \tag{3.42}$$

*and*

$$|b(x)| \leq (1 + |x|)^{-\max(\mathcal{M},s+3)}. \tag{3.43}$$

*For every $Q \in \mathcal{Q}_j$, we set*

$$B_Q(x) := 2^{-j/2} \sum_{\nu=0}^{\infty} \hat{b}(2\pi\nu 2^{-j})\big(e^{2\pi i\nu(x-x_Q)} + e^{2\pi i\nu(x-x_{Q^*})}\big), \text{ when } j \geq 0 \tag{3.44}$$

*and $B_Q(x) := 1$ when $j = -1$.*

*We say that $\{B_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth analysis molecules for $F^s_{pq}$.*

**Remark 3.11.** *(i) In the case of $B^s_{pq}$ spaces one can also allow $p = \infty$. Recall that for these spaces $\mathcal{J} = 1/\min(1, p)$.*

*(ii) The family $\{G_Q\}_{Q\in\mathcal{Q}}$ is a constant multiple of a family of smooth analysis molecules for both $F^s_{pq}$ and $B^s_{pq}$ for every triple of indices $(s, p, q)$.*

With a similar proof as for Theorem 3.8, we obtain that:

**Theorem 3.12.** *(Smooth molecular analysis). Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$. Then there exists a constant $c > 0$ such that if $\{B_Q\}_{Q\in\mathcal{Q}}$ is a family of smooth analysis molecules for $F^s_{pq}$ then*

$$\|\langle f, B_Q\rangle\|_{f^s_{pq}} \leq c\|f\|_{F^s_{pq}}, \text{ for every } f \in F^s_{pq}.$$

The same result holds true for families of smooth analysis molecules for $B^s_{pq}$, where we also may include the case $p = \infty$.

We just mention that the proof is based naturally on the following Lemma.

**Lemma 3.13.** *Let $s \in \mathbb{R}$, $p \in (0,\infty)$ and $q \in (0,\infty]$ (or $p \in (0,\infty]$) and $\{B_Q\}_{Q\in\mathcal{Q}}$ be a family of smooth analysis molecules for $F^s_{pq}$ (or $B^s_{pq}$). Then the operator $A$ given by the matrix*

$$a_{QP} := \langle G_P, B_Q\rangle, \ Q, P \in \mathcal{Q}$$

*is almost diagonal.*

The proof of this Lemma is similar to the one of Lemma 3.9 and we leave it for the reader.

## 4. Fourier multipliers

An extremely well studied class of operators acting on distributions is the class of Fourier multipliers. On $\mathbb{R}^n$ a multiplier or a symbol is a complex-valued function $m : \mathbb{R}^n \to \mathbb{C}$ and the associated Fourier multiplier is the operator $T_m$ defined as the multiplication by $m(\xi)$ in the frequency space, precisely:

$$\widehat{T_m f}(\xi) := m(\xi)\hat{f}(\xi), \ \xi \in \mathbb{R}^n, \ f \in \mathcal{S}'.$$

We say that $T_m$ is bounded from a quasi-normed space $X$ to a quasi-normed space $Y$ if there exists a constant $c = c_m \geq 0$ such that

$$\|T_m f\|_Y \leq c\|f\|_X, \text{ for every } f \in X.$$

Obviously, when $m$ is an $L^\infty$ function, it turns that $T_m$ is bounded on $L^2$ and $\|T_m\|_{2\to 2} \leq \|m\|_\infty$.

At this Section we will use the molecular decompositions proved in Theorems 3.8 and 3.12 to provide sufficient conditions for the boundedness of Fourier multipliers on holomorphic Besov and Triebel-Lizorkin spaces. Let us pass to the definition of such operators.

4.1. **Fourier multipliers on $\mathbb{T}$.** In the quotient space $\mathbb{T}$, a multiplier is simply a sequence $\{m(n)\}_{n\geq 0}$ of complex numbers with the associated *Fourier multiplier* is the operator $T_m$ defined as:

$$(T_m\phi)(x) := \sum_{n=0}^{\infty} m(n)\hat{\phi}(n)e^{2\pi inx}, \quad x \in [0,1), \quad \phi \in \mathcal{D}_+. \tag{4.45}$$

The definition (4.45) extends to distributions by duality. A systematic study of Fourier multipliers on $\mathbb{T}$ (and more generally on $\mathbb{T}^n$) can be found in Grafakos' book [11, Chapter 4].

In this case, when $m \in \ell^\infty$, it turns that $T_m$ is bounded on $L^2$ and $\|T_m\|_{2\to 2} \leq \|m\|_\infty$. For boundedness between spaces other than $L^2$, it is often advantageous to view the sequence $\{m(n)\}_n$ as a sampling sequence for a continuous function $m : \mathbb{R}_+ \to \mathbb{R}$. Then imposing a certain smoothness level and bounds on derivatives of $m$, one can often derive the wanted boundedness results. Following this point of view, we define the multiplier class:

**Definition 4.1.** *Let $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$. We say that a function $m$ belongs to the multiplier class $\mathbf{M}(\alpha, k)$ when $m \in \mathcal{C}^k(\mathbb{R}_+)$ and*

$$|m^{(r)}(\xi)| \leq c_r(1+\xi)^{\alpha-r}, \quad \textit{for every } \xi \in \mathbb{R}_+,\ 0 \leq r \leq k. \tag{4.46}$$

When $m \in \mathbf{M}(\alpha, k)$, Fourier multiplier $T_m$ defined as in (4.45), is well-defined and bounded in the space of test functions. Indeed, let $\rho, n \geq 0$, then by (4.46) and (2.1) we obtain that

$$(1+n)^\rho |m(n)\hat{\phi}(n)| \leq c_0 \mathcal{P}_{\rho+|\alpha|}(\phi), \quad \text{for every } \phi \in \mathcal{D}_+.$$

By duality we extend the action of $T_m$ to the space $\mathcal{D}'_+$ of distributions.

4.2. **The main Fourier multipliers' result.** We are now ready to prove our main multiplier result:

**Theorem 4.2.** *Let $s, \alpha \in \mathbb{R}$, $0 < q \leq \infty$, and let $k \in \mathbb{N}$ be such that $k > \mathcal{J}$. Consider a multiplier $m \in \mathbf{M}(\alpha, k)$.*

*(i) If $0 < p \leq \infty$, then the Fourier multiplier $T_m$ is bounded from $B^{s+\alpha}_{pq}$ to $B^s_{pq}$.*

*(ii) If $0 < p < \infty$, then the Fourier multiplier $T_m$ is bounded from $F^{s+\alpha}_{pq}$ to $F^s_{pq}$.*

*Proof.* We will only work out the details for claim (ii). Claim (i) can be treated in exactly the same way.

Let $f$ be a distribution belonging to $F^{s+\alpha}_{pq}$. By (2.18), $f$ can be decomposed as $f = \sum_{P\in\mathcal{Q}} \langle f, G_P\rangle G_P$. Then we have the action

$$T_m f = \sum_{P\in\mathcal{Q}} \langle f, G_P\rangle T_m G_P. \tag{4.47}$$

On the other hand since $T_m f \in \mathcal{D}'_+$ we have by (2.18),

$$T_m f = \sum_{Q\in\mathcal{Q}} \langle T_m f, G_Q\rangle G_Q, \tag{4.48}$$

therefore we turn to study the sequence $\{\langle T_m f, G_Q\rangle\}$. We observe firstly that

$$\|\langle T_m f, G_Q\rangle\|_{f^s_{pq}} = \Big\| \sum_{P\in\mathcal{Q}} \langle f, G_P\rangle \langle T_m G_p, G_Q\rangle \Big\|_{f^s_{pq}}. \tag{4.49}$$

Define $A$ to be the operator acting on the space of the sequences with matrix

$$\alpha_{QP} := \langle T_m G_P, G_Q\rangle, \quad \text{for every } Q, P \in \mathcal{Q}.$$

We then have that

$$\sum_{P\in\mathcal{Q}} \langle f, G_P\rangle\langle T_m G_P, G_Q\rangle = A\big(\langle f, G_P\rangle\big). \tag{4.50}$$

Thus we need to estimate $\alpha_{QP}$.

Let $Q \in \mathcal{Q}_j,\ P \in \mathcal{Q}_\mu$. We assume that $j \geq \mu \geq 2$; all the other cases are easier or similar. By (2.15) and (4.45), we have the expression

$$\begin{aligned}
\alpha_{QP} &= 2^{-(\mu+j)/2}\sum_{n=0}^{\infty} m(n)\hat{\psi}(2\pi n 2^{-\mu})\overline{\hat{\psi}(2\pi n 2^{-j})}e^{2\pi i n(x_P - x_Q)}\\
&\quad + 2^{-(\mu+j)/2}\sum_{n=0}^{\infty} m(n)\hat{\psi}(2\pi n 2^{-\mu})\overline{\hat{\psi}(2\pi n 2^{-j})}e^{2\pi i n(x_{P^*} - x_Q)}\\
&\quad + 2^{-(\mu+j)/2}\sum_{n=0}^{\infty} m(n)\hat{\psi}(2\pi n 2^{-\mu})\overline{\hat{\psi}(2\pi n 2^{-j})}e^{2\pi i n(x_P - x_{Q^*})}\\
&\quad + 2^{-(\mu+j)/2}\sum_{n=0}^{\infty} m(n)\hat{\psi}(2\pi n 2^{-\mu})\overline{\hat{\psi}(2\pi n 2^{-j})}e^{2\pi i n(x_{P^*} - x_{Q^*})}\\
&=: a_1 + \cdots + a_4. \qquad (4.51)
\end{aligned}$$

Note that $\alpha_{QP} = 0$ when $|j-\mu| \geq 3$, so we are restricted to $j \sim \mu$.

We will estimate the quantity $a_1$. Similar arguments can be use to obtain estimates for $a_2, a_3$ and $a_4$.

We set $\hat{h}(\xi) := \widehat{h_{j,\mu}}(\xi) := m\big(\frac{2^\mu \xi}{2\pi}\big)\hat{\psi}(\xi)\overline{\hat{\psi}(2^{\mu-j}\xi)}$. Thus,

$$m(n)\hat{\psi}(2\pi n 2^{-\mu})\overline{\hat{\psi}(2\pi n 2^{-j})} = \hat{h}(2\pi n 2^{-\mu}), \tag{4.52}$$

with $\operatorname{supp}\hat{h} \subset [2\pi/3, 8\pi/3] =: R$. Therefore, $\int_{\mathbb{R}} x^\gamma h(x)dx = 0$ for every $\gamma \leq k$. By (4.51) and (4.52), we get

$$a_1 = 2^{-(\mu+j)/2}\sum_{n=0}^{\infty}\hat{h}(2\pi n 2^{-\mu})e^{2\pi i n(x_P - x_Q)}. \tag{4.53}$$

In the sequel, we follow the approach outlined in [15, Lemma 5.9]. Based on the fact that $\hat{\psi} \in \mathcal{S}(\mathbb{R})$, $\operatorname{supp}\hat{h} \subset R$, $j \sim \mu$, and using the multiplier assumption (4.46), we obtain

$$\begin{aligned}
|\hat{h}^{(r)}(\xi)| &\leq c\sum_{r_1+r_2+r_3=r} 2^{\mu r_1}2^{(\mu-j)r_3}\Big|m^{(r_1)}\Big(\frac{2^\mu\xi}{2\pi}\Big)\Big|\big|\hat{\psi}^{(r_2)}(\xi)\big|\big|\hat{\psi}^{(r_3)}(2^{\mu-j}\xi)\big|\\
&\leq c\sum_{r_1=0}^{r} 2^{\mu r_1}\Big(1+\frac{2^\mu\xi}{2\pi}\Big)^{\alpha-r_1}\mathbb{1}_R(\xi)(1+\xi)^{-2}\\
&\leq c\sum_{r_1=0}^{r} 2^{\mu r_1}2^{\mu(\alpha-r_1)}(1+\xi)^{-2}\\
&\leq c2^{\mu\alpha}(1+\xi)^{-2}. \qquad (4.54)
\end{aligned}$$

By (4.54), we have for every $\nu \le k$,

$$|x|^\nu |h(x)| \le c \int_0^\infty |\hat{h}^{(\nu)}(\xi)| d\xi \le c2^{\mu\alpha} \int_0^\infty (1+\xi)^{-2} d\xi \le c2^{\mu\alpha},$$

so

$$(1+|x|)^k |h(x)| \le c \sum_{\nu=0}^{k} |x|^\nu |h(x)| \le c2^{\mu\alpha},$$

and consequently,

$$|h(x)| \le c2^{\mu\alpha}(1+|x|)^{-k}. \tag{4.55}$$

We denote by $h_\mu(x) := 2^\mu h(2^\mu x)$ and thus $\widehat{h_\mu}(2\pi n) = \hat{h}(2\pi n 2^{-\mu})$. We use the last espression in (4.53), and apply Poisson's summation formula (2.13), to obtain

$$\begin{aligned} a_1 &= 2^{-(\mu+j)/2} \sum_{n=0}^{\infty} \widehat{h_\mu}(2\pi n) e^{2\pi i n(x_P - x_Q)} \\ &= 2^{-(\mu+j)/2} \sum_{n=0}^{\infty} h_\mu(x_P - x_Q + n) \\ &= 2^{(\mu-j)/2} \sum_{n=0}^{\infty} h\big(2^\mu(x_P - x_Q + n)\big). \end{aligned} \tag{4.56}$$

Combining (4.55) with (4.56), and relying on the fact that $j \sim \mu$ and $|x_P - x_Q| \le \frac{1}{2}$, we arrive at

$$\begin{aligned} |a_1| &\le c2^{\mu\alpha} \sum_{n=0}^{\infty} \big(1 + 2^\mu |x_P - x_Q + n|\big)^{-k} \\ &\le c2^{\mu\alpha}\big(1 + 2^\mu |x_P - x_Q|\big)^{-k} \sum_{n=1}^{\infty} n^{-k} \\ &\le c2^{\mu\alpha}\big(1 + 2^\mu |x_P - x_Q|\big)^{-k} \\ &\le c2^{\mu\alpha} \omega_{PQ}(\varepsilon), \end{aligned} \tag{4.57}$$

since $k > [\mathcal{J}] \ge 1$, and we put $\varepsilon := k - \mathcal{J} > 0$.

In a similar fashion, we can prove that (4.57) holds true for $a_2, a_3$ and $a_4$. Therefore, we may conclude that

$$|\alpha_{QP}| \le c|Q|^{-\alpha} \omega_{PQ}(\varepsilon), \text{ for every } Q, P \in \mathcal{Q}.$$

Hence, the operator $B$ with matrix

$$\beta_{QP} := |Q|^\alpha \alpha_{QP}, \text{ for every } Q, P \in \mathcal{Q},$$

is almost diagonal on $f^s_{pq}$.

Combining Proposition 3.1 with (4.49) and (4.50), we conclude that

$$\begin{aligned} \|\langle T_m f, G_Q \rangle\|_{f^s_{pq}} &\le c\|A\big(\langle f, G_P\rangle\big)\|_{f^s_{pq}} = c\|B\big(|Q|^{-\alpha}\langle f, G_P\rangle\big)\|_{f^s_{pq}} \\ &\le c\||Q|^{-\alpha}\langle f, G_P\rangle\|_{f^s_{pq}} = c\|\langle f, G_P\rangle\|_{f^{s+\alpha}_{pq}}, \end{aligned} \tag{4.58}$$

where we used the straightforward observation that $\|a_Q\|_{f^{s+\alpha}_{pq}} \sim \||Q|^{-\alpha} a_Q\|_{f^s_{pq}}$.

By Remark 3.11 the family $\{G_P\}_P$ is (up-to a constant) a family of smooth analysis molecules for $F^{s+\alpha}_{pq}$. By Theorem 3.12 and since $f \in F^{s+\alpha}_{pq}$ we have that

$$\|\langle f, G_P\rangle\|_{f^{s+\alpha}_{pq}} \leq c\|f\|_{F^{s+\alpha}_{pq}} < \infty.$$

Similarly by Remark 3.7, Theorem 3.8, (4.48) and (4.58) we conclude that

$$\|T_m f\|_{F^s_{pq}} \leq c\|\langle T_m f, G_Q\rangle\|_{f^s_{pq}} \leq c\|\langle f, G_P\rangle\|_{f^{s+\alpha}_{pq}} \leq c\|f\|_{F^{s+\alpha}_{pq}}$$

and the proof is complete. □

**Remark 4.3.** *As a natural generalisation of Fourier multipliers one could consider a suitable notion of pseudodifferential operators adapted to the holomorphic setup. However, we notice that one has to be very careful with the definition in order for the associated operators to be invariant on the holomorphic functions even when considering elementary decomposable symbols. We leave this direction for future work.*

## 5. Some final remarks

In this concluding section we will consider some applications of our results. We start by giving a characterization of Besov and Triebel-Lizorkin spaces based on the multiplier result Theorem 4.2.

5.1. **Equivalent norm characterization.** Let $\alpha \in \mathbb{R}$, we define the multiplier

$$m_\alpha(\xi) := (1+\xi)^\alpha \in \mathcal{C}^\infty(\mathbb{R}_+),$$

which belongs to the class $\mathbf{M}(\alpha, k)$ for every $k \in \mathbb{N}$. By Theorem 4.2, $T_{m_\alpha}$ is bounded from $F^{s+a}_{pq}$ to $F^s_{pq}$ for every $s \in \mathbb{R}$, $0 < p < \infty$, $0 < q \leq \infty$ and the same for Besov spaces. But

$$T_{m_\alpha} f(x) = \sum_{n=0}^{\infty} m_\alpha(n)\hat{f}(n)e^{2\pi i n x} = \sum_{n=0}^{\infty}(n+1)^\alpha \hat{f}(n)e^{2\pi i n x} = J^\alpha f(x),$$

so $\|J^\alpha f\|_{F^s_{pq}} \leq c_\alpha \|f\|_{F^{s+\alpha}_{pq}}$ for some $c_\alpha > 0$ and every $f \in F^{s+\alpha}_{pq}$.

On the other hand,

$$\|f\|_{F^{s+\alpha}_{pq}} = \|J^{-\alpha}(J^\alpha f)\|_{F^{s+\alpha}_{pq}} \leq c_{-\alpha}\|J^\alpha f\|_{F^s_{pq}}$$

so we revisit the following known result,

**Corollary 5.1.** *[24, Theorem 2.4] Let $s, \alpha \in \mathbb{R}$ and $0 < q \leq \infty$.*

*(i) If $0 < p \leq \infty$, then $\|J^\alpha f\|_{B^{s-\alpha}_{pq}}$ is an equivalent quasi-norm for $B^s_{pq}$.*

*(ii) If $0 < p < \infty$, then $\|J^\alpha f\|_{F^{s-\alpha}_{pq}}$ is an equivalent quasi-norm for $F^s_{pq}$.*

5.2. **Molecules for Hardy-Sobolev spaces.** As we mentioned earlier, Hardy and Hardy-Sobolev spaces have attracted significant attention from researchers in complex analysis, see [4, 23] and the references therein. Let us consider the Hardy-Sobolev spaces $H^s_p$, for $s \geq 0$. These spaces coincide with Triebel-Lizorkin ones for $q = 2$. Then the values of the important parameters are:

$$\mathcal{J} = \frac{1}{\min(1,p)} \text{ and } K = [s] + 1.$$

Now pick $m \in \mathcal{C}^{[s]+3}(\mathbb{R})$, and $\mathcal{M} > J + 2$, satisfying:

$$|m^{(\gamma)}(x)| \leq (1+|x|)^{-\max(\mathcal{M}, s+3)}, \text{ for every } 0 \leq \gamma \leq [s]+3,$$

and if $0 \leq s \leq \mathcal{J} - 1$,

$$\int_{\mathbb{R}} x^\gamma m(x)dx = 0, \text{ for every } 0 \leq \gamma \leq [\mathcal{J} - s] - 1.$$

Then the family $\{M_Q\}$ defined as in (3.25) is a family of smooth synthesis molecules for $H_p^s,\ s \geq 0$.

We mention here that if $s > \mathcal{J} - 1$, we do not need any vanishing moment condition. Specifically, for Hardy spaces $H_p = H_p^0$, we only ask for

$$\int_{\mathbb{R}} x^\gamma m(x)dx = 0, \text{ for every } 0 \leq \gamma \leq [\mathcal{J}] - 1,$$

the usual vanishing moment condition for $H_p$-atoms. Especially when $p \geq 1$ then we demand only that $\int_{\mathbb{R}} m(x)dx = 0$.

5.3. **Coefficient multipliers.** Fourier multipliers can be extended from the circle to the unit disk. A multiplier in the unit disk is a sequence $\{m(n)\}_{n\geq 0}$ of complex numbers and the associated *coefficient multiplier* is the operator $T_m$ defined as:

$$(T_m f)(z) := \sum_{n=0}^{\infty} m(n)\hat{f}(n)z^n, \quad z \in \mathbb{D}, \quad f \in \mathcal{A}. \tag{5.59}$$

The problem of obtaining boundedness of coefficient multipliers for functions in the complex plane has been the focus of studies for decades. See for example [3, 7, 18, 23] and the references therein.

By extending the classes of multipliers from Definition 4.1 to coefficient multipliers, we easily obtain the following:

(1) For any $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$, a multiplier $m \in \mathbf{M}(\alpha, k)$ induces a well-defined coefficient multiplier $T_m$ from $\mathcal{A}$ to itself.
(2) Theorem 4.2 holds true for coefficient multipliers as well.
(3) In the special case of Hardy-Sobolev spaces, it suffices to require $m \in \mathbf{M}(\alpha, k)$ for $k > \frac{1}{\min(1,p)}$. Then $T_m$ is bounded from $H_p^{s+\alpha}$ to $H_p^s$. In particular, when $\alpha = s = 0$, $T_m$ is bounded on $H_p$.

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